

# RANK-STABLE LIMIT OF COMPLETED MODULI SPACES OF INSTANTONS

JOÃO PAULO SANTOS

**ABSTRACT.** We show that in the rank-stable limit, the inclusion of the moduli space of based instantons over  $S^4$  into a certain completion is a homotopy equivalence. We prove also an analogous result for the moduli space of instantons over  $\mathbb{P}^2$ .

## 1. INTRODUCTION

Given a principal  $SU(r)$  bundle  $P \rightarrow S^4$ , an instanton is a connection  $\nabla$  on  $P$  whose curvature  $F_\nabla$  is a minimum of the Yang-Mills functional  $\int |F_\nabla|^2$ . The moduli space  $\mathfrak{M}(P)$  of instantons based at  $x_0 \in S^4$  is the quotient of the space of instantons by the action of the gauge group of automorphisms of  $P$  which are the identity on  $x_0$ . It depends only on  $r$  and  $c_2(P) = k$  so we represent it by  $\mathfrak{M}_k^r(S^4)$ . In [3], Nakajima introduced a completion  $\overline{\mathfrak{M}}_k^r(S^4)$  of  $\mathfrak{M}_k^r(S^4)$  which is a resolution of singularities of the usual Donaldson-Uhlenbeck completion of  $\mathfrak{M}_k^r(S^4)$ . In this paper we extend this result to instantons on  $\mathbb{P}^2$  and show that, in the limit when  $r \rightarrow \infty$ , the inclusions  $\mathfrak{M} \rightarrow \overline{\mathfrak{M}}$  are homotopy equivalences.

For  $r' > r$  there is an inclusion of pairs  $(\overline{\mathfrak{M}}_k^r, \mathfrak{M}_k^r) \rightarrow (\overline{\mathfrak{M}}_k^{r'}, \mathfrak{M}_k^{r'})$  induced by the inclusion  $SU(r) \rightarrow SU(r')$ . In [2], [4] it was shown that the direct limit  $\mathfrak{M}_k^\infty(X) = \lim_r \mathfrak{M}_k^r(X)$  has the homotopy type of  $BU(k)$  for  $X = S^4$ , and  $BU(k) \times BU(k)$  for  $X = \mathbb{P}^2$ .

## 2. INSTANTONS ON $S^4$

We summarize the monad description of instantons on  $S^4$  and  $\mathbb{P}^2$  and the resolution of singularities of its completion introduced in [3]. Let  $W$  be a  $k$ -dimensional hermitian vector space. Let  $\mathcal{R}_k^r$  be the space of configurations  $(a_1, a_2, b, c)$  with  $a_i \in \text{End}(W)$ ,  $b \in \text{Hom}(\mathbb{C}^r, W)$ ,  $c \in \text{Hom}(W, \mathbb{C}^r)$ , obeying the integrability condition

$$[a_1, a_2] + bc = 0$$

and the perturbed moment map equation

$$[a_1, a_1^*] + [a_2, a_2^*] + bb^* - c^*c = -\zeta$$

for some non-zero real parameter  $\zeta$ . Let  $\tilde{\mathcal{R}}_k^r \subset \mathcal{R}_k^r$  be the subspace of 4-tuples obeying the two non-degeneracy conditions

- (1) There is no proper subspace  $W' \subset W$  such that

$$\text{Im } b \subset W' \text{ and } a_i(W') \subset W' \ (i = 1, 2)$$

- (2) There is no nonempty subspace  $W' \subset W$  such that

$$W' \subset \text{Ker } c \text{ and } a_i(W') \subset W' \ (i = 1, 2)$$

Observe that these conditions are automatically satisfied if both  $b$  and  $c$  have maximal rank. The group  $U(W)$  acts freely on  $\mathcal{R}_k^r$  by

$$g \cdot (a_1, a_2, b, c) = (ga_1g^{-1}, ga_2g^{-1}, gb, cg^{-1})$$

The quotient  $\tilde{\mathcal{R}}_k^r/U(W)$  is isomorphic to the moduli space  $\mathfrak{M}_k^r(S^4)$  and the quotient  $\overline{\mathfrak{M}}_k^r(S^4) = \mathcal{R}/U(W)$  is a resolution of singularities of the Donaldson-Uhlenbeck completion of  $\mathfrak{M}_k^r(S^4)$ .

Now we consider the limit when  $r \rightarrow \infty$ . For  $r' > r$ , the inclusion  $\mathbb{C}^r \rightarrow \mathbb{C}^{r'}$  induces a map  $i_{r,r'} : \mathcal{R}_k^r \rightarrow \mathcal{R}_k^{r'}$ . This map preserves the subspace  $\tilde{\mathcal{R}}$  and descends to the quotient to give a map of pairs

$$(\overline{\mathfrak{M}}_k^r(S^4), \mathfrak{M}_k^r(S^4)) \rightarrow (\overline{\mathfrak{M}}_k^{r'}(S^4), \mathfrak{M}_k^{r'}(S^4))$$

We define  $\mathfrak{M}_k^\infty(S^4) = \varinjlim_r \mathfrak{M}_k^r(S^4)$ ,  $\overline{\mathfrak{M}}_k^\infty(S^4) = \varinjlim_r \overline{\mathfrak{M}}_k^r(S^4)$ .

**Theorem 2.1.** *The inclusion  $j : \mathfrak{M}_k^\infty(S^4) \rightarrow \overline{\mathfrak{M}}_k^\infty(S^4)$  is a homotopy equivalence.*

*Proof.* We first show that the space  $\mathcal{R}_k^\infty$  is contractible. This space is a CW-complex hence it is enough to show that its homotopy groups are trivial, and this will follow if we show that the inclusions  $i_{r,r+2k} : \mathcal{R}_k^r \rightarrow \mathcal{R}_k^{r+2k}$  are null-homotopic. We identify  $\mathbb{C}^{r+2k}$  with  $\mathbb{C}^r \oplus W \oplus W$  via a complex hermitian isomorphism. Then we pick  $\zeta_b, \zeta_c \in \mathbb{R}$  such that  $\zeta_c - \zeta_b = \zeta$  and define a homotopy  $H : \mathcal{R}_k^r \rightarrow \mathcal{R}_k^{r+2k}$  by

$$H_t(a_1, a_2, b, c) = (\sqrt{1-t}a_1, \sqrt{1-t}a_2, b_t, c_t)$$

where

$$b_t = \begin{pmatrix} \sqrt{1-t}b & 0 & \sqrt{t\zeta_b}1 \end{pmatrix} \quad c_t = \begin{pmatrix} \sqrt{1-t}c \\ \sqrt{t\zeta_c}1 \\ 0 \end{pmatrix}$$

It is a direct verification that  $H$  is a well defined homotopy between  $i_{r,r+2k}$  and a constant map. Furthermore, since  $b_t, c_t$  both have maximal rank for  $t \neq 0$ , the restriction of  $H$  to  $\tilde{\mathcal{R}}_k^r$  defines a homotopy  $H : \tilde{\mathcal{R}}_k^r \rightarrow \tilde{\mathcal{R}}_k^{r+2k}$ . Hence we also conclude that  $\tilde{\mathcal{R}}_k^\infty$  is contractible.

We now observe that the principal  $U$ -bundle  $\tilde{\mathcal{R}}_k^\infty \rightarrow \mathfrak{M}_k^\infty(S^4)$  is the pullback  $j^*\mathcal{R}_k^\infty$  of the bundle  $\mathcal{R}_k^\infty \rightarrow \overline{\mathfrak{M}}_k^\infty$ . Applying the five lemma to the long exact sequence of homotopy groups associated with these principal bundles, it follows that  $j$  induces isomorphisms on all homotopy groups, hence it is a homotopy equivalence.  $\square$

### 3. INSTANTONS ON $\mathbb{P}^2$

We now look at instantons on  $\mathbb{P}^2$ . We begin by sketching the monad description introduced in [1]. Let  $W_0, W_1$  be  $k$ -dimensional hermitian vector spaces. Let  $\mathcal{C}_k^r$  be the space of configurations  $m = (a_1, a_2, d, b, c)$  where  $a_i \in \text{Hom}(W_1, W_0)$ ,  $d \in \text{Hom}(W_0, W_1)$ ,  $b \in \text{Hom}(\mathbb{C}^r, W_0)$ ,  $c \in \text{Hom}(W_1, \mathbb{C}^r)$ , such that  $a_1(W_1) + a_2(W_1) + b(\mathbb{C}^r) = W_0$ , obeying the integrability condition

$$a_1da_2 - a_2da_1 + bc = 0.$$

Let  $\tilde{\mathcal{C}}_k^r \subset \mathcal{C}_k^r$  be the subspace of configurations obeying the non-degeneracy conditions

- (1) There are no proper subspaces  $W'_0 \subset W_0$  and  $W'_1 \subset W_1$  such that  $\dim W'_0 = \dim W'_1$ ,  $\text{Im } b \subset W'_0$ ,  $d(W'_0) \subset W'_1$  and  $a_i(W'_1) \subset W'_0$  ( $i = 1, 2$ )
- (2) There are no nonempty subspaces  $W'_0 \subset W_0$  and  $W'_1 \subset W_1$  such that  $\dim W'_0 = \dim W'_1$ ,  $W'_1 \subset \text{Ker } c$ ,  $d(W'_0) \subset W'_1$  and  $a_i(W'_1) \subset W'_0$  ( $i = 1, 2$ )

These conditions are automatically satisfied if  $b$  and  $c$  have maximal rank. The group  $Gl(W_0) \times Gl(W_1)$  acts on  $\mathcal{C}_k^r$  by

$$(g_0, g_1) \cdot (a_1, a_2, d, b, c) = (g_0 a_1 g_1^{-1}, g_0 a_2 g_1^{-1}, g_1 d g_0^{-1}, g_0 b, c g_1^{-1})$$

The restriction of this action to  $\tilde{\mathcal{C}}_k^r$  is free and the quotient is isomorphic to the moduli space  $\mathfrak{M}_k^r(\mathbb{P}^2)$ .

Now let  $\mathcal{R}_k^r \subset \mathcal{C}_k^r$  be the subspace of configurations obeying the moment map equations

$$\begin{cases} a_1 a_1^* + a_2 a_2^* + b b^* = 1 \\ a_1^*(1 + d^* d) a_1 + a_2^*(1 + d^* d) a_2 + c^* c = 1 + d d^* \end{cases}$$

Using the first equation, we can write the second equation in the sometimes more useful form

$$[da_1, (da_1)^*] + [da_1, (da_1)^*] - a_1^* a_1 - a_2^* a_2 + db(db)^* - c^* c = -1$$

Let  $\tilde{\mathcal{R}}_k^r = \tilde{\mathcal{C}}_k^r \cap \mathcal{R}_k^r$ . The  $Gl(W_0) \times Gl(W_1)$  action on  $\mathcal{C}_k^r$  induces a free  $U(W_0) \times U(W_1)$  action on  $\tilde{\mathcal{R}}_k^r$  and the quotient is also isomorphic to  $\mathfrak{M}_k^r(\mathbb{P}^2)$  (see [1]).

We now perturb the moment map equations by introducing a non-integer positive parameter  $\zeta \in \mathbb{R}^+ \setminus \mathbb{Z}$ :

$$(1) \quad \begin{cases} a_1 a_1^* + a_2 a_2^* + b b^* = 1 \\ a_1^*(1 + d^* d) a_1 + a_2^*(1 + d^* d) a_2 + c^* c = \zeta + d d^* \end{cases}$$

Again, we can substitute the second equation by

$$(2) \quad [da_1, (da_1)^*] + [da_1, (da_1)^*] - a_1^* a_1 - a_2^* a_2 + db(db)^* - c^* c = -\zeta$$

We let  $\mathcal{R}_\zeta \subset \mathcal{C}$  denote the subspace of solutions to these equations.

**Theorem 3.1.** *The  $U(W_0) \times U(W_1)$  action on  $\mathcal{R}_\zeta$  is free.*

*Proof.* Suppose  $(g_0, g_1)$  stabilizes  $(a_1, a_2, d, b, c)$ . Let  $W_i(\lambda) \subset W_i$  denote the  $\lambda$ -eigenspace of  $g_i$ . Then

$$d(W_0(\lambda)) \subset W_1(\lambda), \quad a_i(W_1(\lambda)) \subset W_0(\lambda) \quad (i = 1, 2)$$

and, for  $\lambda \neq 1$ ,

$$W_0(\lambda) \subset \text{Ker } b^* \text{ and } W_1(\lambda) \subset \text{Ker } c$$

Hence, restricting the perturbed moment map equations (1) to  $W_i(\lambda)$ .  $\lambda \neq 1$ , and substituting (2) for the second equation we get

$$\begin{cases} a_1 a_1^* + a_2 a_2^* = 1 \\ [da_1, (da_1)^*] + [da_1, (da_1)^*] - a_1^* a_1 - a_2^* a_2 = -\zeta \end{cases}$$

Taking the trace, since  $\text{Tr}(a_1 a_1^* + a_2 a_2^*) = \text{Tr}(a_1^* a_1 + a_2^* a_2)$  we get  $\dim W_0(\lambda) = \zeta \dim W_1(\lambda)$ . This is impossible unless  $W_0(\lambda) = W_1(\lambda) = 0$  for any  $\lambda \neq 1$  which implies that  $(g_0, g_1) = (1, 1)$ .  $\square$

Let  $\overline{\mathfrak{M}}_k^r(\mathbb{P}^2)$  be the quotient of  $\mathcal{R}_{\zeta,k}^r$  by the  $U(W_0) \times U(W_1)$  action. We again have maps of pairs

$$\left(\overline{\mathfrak{M}}_k^r(\mathbb{P}^2), \mathfrak{M}_k^r(\mathbb{P}^2)\right) \rightarrow \left(\overline{\mathfrak{M}}_k^{r'}(\mathbb{P}^2), \mathfrak{M}_k^{r'}(\mathbb{P}^2)\right)$$

and we define  $\mathfrak{M}_k^\infty(S^4) = \varinjlim_r \mathfrak{M}_k^r(S^4)$ ,  $\overline{\mathfrak{M}}_k^\infty(S^4) = \varinjlim_r \overline{\mathfrak{M}}_k^r(S^4)$ .

**Theorem 3.2.** *The inclusion  $j : \mathfrak{M}_k^\infty(\mathbb{P}^2) \rightarrow \overline{\mathfrak{M}}_k^\infty(\mathbb{P}^2)$  is a homotopy equivalence.*

*Proof.* The proof follows the same lines as the one for  $S^4$ . We will show that the inclusions  $i_{r,r+3k} : \mathcal{R}_k^r \rightarrow \mathcal{R}_k^{r+2k}$  are null-homotopic. First we identify  $\mathbb{C}^{r+3k}$  with  $\mathbb{C}^r \oplus W_0 \oplus W_0 \oplus W_1$  via a complex hermitian isomorphism. Then we define a homotopy  $H : \mathcal{R}_k^r \rightarrow \mathcal{R}_k^{r+2k}$  by

$$H_t(a_1, a_2, d, b, c) = (\sqrt{1-t}a_1, \sqrt{1-t}a_2, d, b_t, c_t)$$

where

$$b_t = \begin{pmatrix} \sqrt{1-t}b & 0 & \sqrt{t}1 & 0 \end{pmatrix} \quad c_t = \begin{pmatrix} \sqrt{1-t}c \\ \sqrt{t}d^* \\ 0 \\ \sqrt{t\zeta}1 \end{pmatrix}$$

It is a direct verification that  $H$  is a well defined homotopy between  $i_{r,r+2k}$  and the map  $f : \mathcal{R}_k^r \rightarrow \tilde{\mathcal{R}}_k^{r+3k}$  given by  $f(a_1, a_2, d, b, c) = (0, 0, d, b_1, c_1)$ . Furthermore, the restriction of  $H$  to  $\tilde{\mathcal{R}}_k^r$  defines a homotopy  $H : \tilde{\mathcal{R}}_k^r \rightarrow \tilde{\mathcal{R}}_k^{r+2k}$ . To conclude the proof we follow  $H$  with another homotopy

$$F_t(a_1, a_2, d, b, c) = (0, 0, (1-t)d, b_1, C_t)$$

where

$$C_t = (0 \quad (1-t)d^* \quad 0 \quad \sqrt{\zeta}1)^T$$

$F$  is then a homotopy between  $f$  and a constant map, which finishes the proof.  $\square$

## REFERENCES

1. A. King, *Instantons and holomorphic bundles on the blown-up plane*, Ph.D. thesis, Worcester College, Oxford, 1989.
2. F. Kirwan, *Geometric invariant theory and the Atiyah-Jones conjecture*, The Sophus Lie Memorial Conference (Oslo, 1992), Scand. Univ. Press, Oslo, 1994, pp. 161–186.
3. H. Nakajima, *Resolutions of moduli spaces of ideal instantons on  $\mathbb{R}^4$* , Topology, geometry and field theory, World Sci. Publishing, River Edge, NJ, 1994, pp. 129–136.
4. M. Sanders, *Classifying spaces and Dirac operators coupled to instantons*, Trans. Amer. Math. Soc. **347** (1995), no. 10, 4037–4072.